A variance-scaled Turán inequality at the first descent of a Poisson–binomial mass function[*]

Brett Reynolds[†]

Abstract

Let W be a finite sum of independent Bernoulli summands with probability mass function (pmf) $f = (f_k)$, where $f_k = \mathbb{P}(W = k)$, and variance $V = \mathrm{Var}(W) \geq 1$. If D is the first-descent index of this pmf, we prove

$$V\left(1 - \frac{f_{D-1}f_{D+1}}{f_D^2}\right) \geq \frac{1}{4}.$$

This gives a variance-scaled bound for the normalized Turán deficit at D and yields $f_{D+r}/f_D \leq \exp(-r/(4V))$ for every support index $D+r$ with $r \geq 1$. An explicit family of finite binomial laws shows that no universal constant larger than $1/3$ is possible. The proof derives left and right normalized-mass lower bounds about the rightmost modal index from cubic mass inequalities, combines them with a lattice maximal-mass bound, and proves the remaining scalar inequality directly outside a compact range and by exact Bernstein expansions with strictly positive rational coefficients within it.

Keywords: Poisson–binomial law; Bernoulli sum; probability mass function; log-concavity; Turán inequality; modal index; computer-assisted proof.
MSC2020 subject classifications: Primary 60E15, Secondary 60C05; 05A20.

1 Introduction and main results

Let

$$W = \sum_{i=1}^{n} B_i, \qquad B_i \sim \mathrm{Bernoulli}(p_i),$$

where the summands are independent. The law of W is Poisson–binomial; see [11] for a recent survey.

Bernoulli summands with $p_i = 0$ may be deleted, whereas those with $p_i = 1$ contribute a deterministic shift. Put $I = \{i : 0 < p_i < 1\}$, $s = \#\{i : p_i = 1\}$, and $W^\circ = \sum_{i \in I} B_i$. Then $W = s + W^\circ$ and $\mathrm{Var}(W^\circ) = \mathrm{Var}(W)$. Let $h = (h_k)$ be the pmf of W°, where $h_k = \mathbb{P}(W^\circ = k)$. Then $\mathbb{P}(W = s + k) = h_k$; hence translation shifts the first-descent index by s and preserves the corresponding adjacent mass ratios and normalized Turán deficits wherever defined.

[*] Generative AI systems assisted with this work; see the disclosure and audit status at the end.

[†] Humber College, Toronto, Canada. E-mail: `brett.reynolds@humber.ca`

Assume $\mathrm{Var}(W) \geq 1$. Then I is nonempty. We henceforth replace W by W°, relabel $|I|$ as n, and assume that every success probability satisfies $0 < p_i < 1$. Put

$$f_k = \mathbb{P}(W = k), \qquad V = \mathrm{Var}(W) = \sum_{i=1}^{n} p_i(1 - p_i),$$

and call $(f_k)_{k=0}^n$ the probability mass function (pmf) of W. Extend the pmf by $f_{-1} = f_{n+1} = 0$. The probability-generating polynomial $\sum_{k=0}^n f_k x^k$ has only negative real zeros, so Newton's inequalities make the pmf log-concave [10, equation (1)]. For $0 \leq k \leq n$, define the normalized Turán deficit

$$\delta_k := 1 - \frac{f_{k-1} f_{k+1}}{f_k^2}. \tag{1.1}$$

We have $0 \leq \delta_k \leq 1$ and $\delta_0 = \delta_n = 1$.

Let

$$D := \min\{1 \leq k \leq n : f_k < f_{k-1}\}. \tag{1.2}$$

This set is nonempty. Indeed,

$$\frac{f_n}{f_{n-1}} = \left(\sum_{i=1}^{n} \frac{1 - p_i}{p_i} \right)^{-1}.$$

If $f_n \geq f_{n-1}$, then the sum on the right is at most one, whereas

$$V = \sum_{i=1}^{n} p_i^2 \frac{1 - p_i}{p_i} < \sum_{i=1}^{n} \frac{1 - p_i}{p_i} \leq 1,$$

a contradiction. By log-concavity, $c := D - 1$ is the rightmost modal index.

Ordinary log-concavity gives only $\delta_D \geq 0$. The sharper $\mathrm{ULC}(n)$ estimate scales with the number of summands, which can grow while the variance remains fixed. Darroch's localization theorem controls modal-index location; variance-scaled maximal-mass bounds control the maximal mass; neither controls the normalized Turán deficit. The cubic mass inequalities of Hillion and Johnson supply a recurrence for that deficit but don't compare it with variance [3, 4, 7].

The variance of the Bernoulli summand B_i is $p_i(1 - p_i)$, which tends to zero as p_i approaches 0 or 1. Thus the natural local question is whether $V\delta_D$ is uniformly bounded away from zero. The answer is yes.

Theorem 1.1. *Under the standing notation above, suppose $V \geq 1$ and let D be defined by* (1.2)*. Then*

$$V\delta_D = V \left(1 - \frac{f_{D-1} f_{D+1}}{f_D^2} \right) \geq \frac{1}{4}. \tag{1.3}$$

A positive universal bound without a lower variance threshold is impossible. If $W \sim \mathrm{Bernoulli}(p)$ with $0 < p < 1/2$, then $D = 1$, $\delta_D = 1$, and $V\delta_D = p(1 - p) \to 0$ as $p \downarrow 0$.

Let $c_\star$ denote the infimum of $V\delta_D$ over all finite Poisson–binomial laws with $V \geq 1$, where D and δ_D are defined from each law's pmf as above. The theorem and the next proposition give

$$\frac{1}{4} \leq c_\star \leq \frac{1}{3}. \tag{1.4}$$

Proposition 1.2. *For every $n \geq 5$, there exists a binomial law of variance one whose pmf has first-descent index $D = 2$ and for which*

$$\delta_D = \frac{n+1}{3(n-1)}.$$

This gives $c_\star \leq 1/3$.

Proof. Set

$$p_n = \frac{1-\sqrt{1-4/n}}{2}, \qquad W_n \sim \mathrm{Bin}(n, p_n).$$

Then $np_n(1-p_n) = 1$. Write $f = (f_k)$ for the pmf of W_n and put $q_k = f_k/f_{k-1}$. Then

$$q_k = \frac{n-k+1}{k}\frac{p_n}{1-p_n}.$$

Since $p_n(1-p_n) = 1/n$ and $p_n < 1$, we have $p_n > 1/n$ and hence $q_1 > 1$. Also $p_n < 2/(n+1)$ for $n \geq 5$: on $[0, 1/2]$ the map $x \mapsto x(1-x)$ is increasing, and

$$\frac{2}{n+1}\left(1 - \frac{2}{n+1}\right) = \frac{2(n-1)}{(n+1)^2} > \frac{1}{n}.$$

So $q_2 < 1$ and $D = 2$. Finally,

$$\delta_2 = 1 - \frac{q_3}{q_2} = 1 - \frac{2(n-2)}{3(n-1)} = \frac{n+1}{3(n-1)} \longrightarrow \frac{1}{3}.$$

$\square$

Proposition 1.2 supplies a limiting admissible family but no extremality argument. The lower-bound proof also uses several relaxations. It first derives explicit left and right normalized-mass lower bounds. When the scalar H introduced in Section 4 satisfies $H \geq 4$, the proof replaces each right normalized-mass lower bound by a smaller symmetric normalized-mass lower bound before applying the general maximal-mass bound. The inequalities defining these normalized-mass lower bounds and the maximal-mass bound need not be equalities simultaneously. The argument establishes (1.4), and we make no conjecture about which, if either, endpoint equals $c_\star$.

Theorem 1.1 also bounds the adjacent-ratio drop $1 - f_{D+1}/f_D$; log-concavity then bounds f_{D+r}/f_D at every support index $D + r > D$. Complementation followed by translation extends these conclusions to differences of independent Bernoulli sums.

Corollary 1.3. *Under the hypotheses of Theorem 1.1,*

$$V\left(1 - \frac{f_{D+1}}{f_D}\right) \geq \frac{1}{4}. \tag{1.5}$$

Writing $\rho_V = 1 - (4V)^{-1}$, we also have

$$\frac{f_{D+r}}{f_D} \leq \rho_V^r \leq \exp\left(-\frac{r}{4V}\right), \qquad 1 \leq r \leq n - D. \tag{1.6}$$

More generally, let X and Y be finite Bernoulli sums, with all summands in the two sums mutually independent, and put $Z = X - Y$. Write $g = (g_j)$ for the pmf of Z, so $g_j = \mathbb{P}(Z = j)$, and assume $V_Z = \mathrm{Var}(Z) \geq 1$. Define $d := \min\{j \in \mathbb{Z} : g_j > 0 \text{ and } g_j < g_{j-1}\}$. Complementation followed by translation to a Bernoulli sum shows that this index is well defined. If $u = \max\{j : g_j > 0\}$ and $\rho_Z = 1 - (4V_Z)^{-1}$, then

$$V_Z\left(1 - \frac{g_{d-1}g_{d+1}}{g_d^2}\right) \geq \frac{1}{4}, \qquad V_Z\left(1 - \frac{g_{d+1}}{g_d}\right) \geq \frac{1}{4}.$$

We also have

$$\frac{g_{d+r}}{g_d} \leq \rho_Z^r \leq \exp\left(-\frac{r}{4V_Z}\right), \qquad 1 \leq r \leq u - d.$$

Proof. Write $\eta_- = f_D/f_{D-1} < 1$ and $\eta_+ = f_{D+1}/f_D$. Then

$$1 - \eta_+ \geq 1 - \frac{\eta_+}{\eta_-} = \delta_D,$$

proving (1.5). Log-concavity makes the adjacent mass ratios nonincreasing. Hence, for $1 \leq j \leq r$,

$$\frac{f_{D+j}}{f_{D+j-1}} \leq \frac{f_{D+1}}{f_D} \leq \rho_V.$$

Multiplication gives the first inequality in (1.6), and $1 - x \leq e^{-x}$ gives the second.

Write $Y = \sum_{j=1}^m Y_j$. Then

$$Z + m = X + \sum_{j=1}^{m}(1 - Y_j)$$

is a finite Bernoulli sum. Translation preserves the variance and the corresponding adjacent mass ratios and normalized Turán deficits while shifting the first-descent index, so the same argument gives all three signed estimates. □

Equation (1.6) gives a variance-scaled geometric upper bound for the normalized mass f_{D+r}/f_D at every support index $D + r$ with $1 \leq r \leq n - D$. The standard $\mathrm{ULC}(n)$ estimate for Poisson–binomial pmfs alone doesn't recover this conclusion:

$$\delta_k \geq \frac{n+1}{(k+1)(n-k+1)}; \tag{1.7}$$

see [7, Definition 3.11 and equation (41)] and, for a recent treatment via relative log-concavity, [9, the discussion following Definition 1.3]. This bound is indexed by the number of summands rather than by the variance V of their sum.

For example, take m success probabilities equal to ε_m and m equal to $1 - \varepsilon_m$, where $2m\varepsilon_m(1-\varepsilon_m) = 1$. The resulting pmf is symmetric and strictly log-concave. The law has $V = 1$, and the pmf has first-descent index $D = m + 1$, but the right-hand side of (1.7) at D is

$$\frac{2m+1}{m(m+2)} \longrightarrow 0.$$

Even at fixed variance, the $\mathrm{ULC}(n)$ lower bound at the first-descent index tends to zero.

Existing quantitative results control modal-index location, adjacent mass ratios, or individual masses rather than the variance-scaled normalized Turán deficit at the first-descent index. Darroch shows that a modal index lies within one unit of the mean [4]. Pitman bounds adjacent mass ratios through exponentially tilted real-rooted laws [10, equations (20)–(21)], and Dümbgen and Wellner prove strict decrease of the index-weighted mass ratios $(k+1)f_{k+1}/f_k$ [5, Proposition 2].

Gnedin extends the mean–mode rule to extended Bernoulli sums [6, Theorem 2]. A three-mass peak-skewness statistic measures the Bernoulli perturbation needed to change the leading mode [6, Sections 4.2 and 7], rather than the normalized Turán deficit at the first-descent index considered here.

Johnson's c-log-concavity bound also controls δ_k, but its scale depends on the success probabilities:

$$\delta_k \geq \frac{f_{k+1}}{f_k} \left(\sum_i \frac{p_i}{1 - p_i} \right)^{-1};$$

see [8, Definition 1.2, Lemma 5.1, and Example 5.2(2)]. In the variance-one symmetric family above, the reciprocal of the odds sum tends to zero. At the first-descent index, the adjacent mass ratio is at most one, so the right-hand side of Johnson's bound also tends to zero.

The uniform maximal-mass bound of Baillon, Cominetti, and Vaisman and the lattice maximal-mass bound of Bobkov, Marsiglietti, and Melbourne control one mass rather than the normalized Turán deficit in (1.1) [2, 3]. We aren't aware of an earlier universal constant $c > 0$ for which $V\delta_D \geq c$ at the first-descent index of the pmf of every finite Poisson–binomial law with $V \geq 1$.

For an integer-valued log-concave random variable X with maximal mass M, Aravinda's sharp min-entropy/variance inequality gives [1, Theorem 1.4]

$$M^{-2} \geq 1 + \mathrm{Var}(X).$$

This bounds the maximal mass from above. Its direction is opposite to the input needed here: (3.3) gives $M \geq (1 + 12V)^{-1/2}$, a lower bound on the maximal mass. Aravinda's result complements that step and reinforces the distinction between variance control of one atom and variance control of a local normalized Turán deficit.

The proof uses two external results. The cubic mass inequalities of Hillion and Johnson yield a recurrence; iteration yields explicit left and right normalized-mass lower bounds about the rightmost modal index. The pairwise variance identity turns those bounds into a variance estimate; combining that estimate with the maximal-mass bound of Bobkov, Marsiglietti, and Melbourne reduces the theorem to a scalar inequality. A direct symbolic argument proves that inequality for $H \geq 16$; exact Bernstein expansions with strictly positive rational coefficients prove it for $3 < H \leq 16$, without floating-point sampling.

2 The recurrence for normalized Turán deficits

For $1 \leq k \leq n$, put

$$q_k = \frac{f_k}{f_{k-1}}.$$

Log-concavity says that (q_k) is nonincreasing, and for $1 \leq k \leq n-1$,

$$\delta_k = 1 - \frac{q_{k+1}}{q_k}. \tag{2.1}$$

Hillion and Johnson's Theorem A.2 and Corollary A.3 state, for every $k \in \mathbb{Z}$, the cubic mass inequalities [7, (78)–(79)]

$$(f_{k-1}^2 - f_{k-2}f_k)f_{k+1} \leq f_{k-1}(f_k^2 - f_{k-1}f_{k+1}), \tag{2.2}$$
$$(f_{k+1}^2 - f_k f_{k+2})f_{k-1} \leq f_{k+1}(f_k^2 - f_{k-1}f_{k+1}). \tag{2.3}$$

Their all-integer statement understands the pmf as zero beyond its support, so these inequalities include the cases adjacent to a support boundary. The identities

$$\begin{aligned} f_{k-1}^2 - f_{k-2}f_k &= f_{k-1}^2\delta_{k-1}, \\ f_k^2 - f_{k-1}f_{k+1} &= f_k^2\delta_k, \\ f_{k+1}^2 - f_k f_{k+2} &= f_{k+1}^2\delta_{k+1} \end{aligned}$$

show the normalization explicitly. For $1 \leq k \leq n-1$, dividing (2.2) by $f_{k-1}f_k^2$ and (2.3) by $f_{k+1}f_k^2$, respectively, gives the recurrence for normalized Turán deficits

$$\delta_{k-1}(1 - \delta_k) \leq \delta_k, \qquad \delta_{k+1}(1 - \delta_k) \leq \delta_k. \tag{2.4}$$

In particular, this includes relations adjacent to the boundary, such as $\delta_n(1 - \delta_{n-1}) \leq \delta_{n-1}$.

Set $\delta = \delta_D$. The boundary values show that $\delta > 0$: if $\delta = 0$, (2.4) forces both adjacent values of δ_k to vanish; repeated application then forces every δ_k to vanish, contradicting $\delta_0 = \delta_n = 1$. If $\delta \geq 1/4$, Theorem 1.1 follows from $V \geq 1$, so it remains to consider

$$0 < \delta < \frac{1}{4}. \tag{2.5}$$

In this case D lies strictly inside the support: $D \geq 1$ by definition, whereas $\delta_D < 1 = \delta_n$ forces $D < n$.

Iterating the map $x \mapsto x/(1-x)$ in (2.4) gives, for either sign and every integer $r \geq 1$,

$$\delta_{D\pm r} \leq \frac{\delta}{1 - r\delta} \qquad \text{provided } D \pm r \in \{0, \dots, n\} \text{ and } (r+1)\delta < 1. \tag{2.6}$$

The same bound excludes an endpoint index of the support in this range: otherwise the left-hand side would equal one, while the right-hand side is strictly less than one.

Define

$$K = \max\{r \in \mathbb{Z}_{\geq 1} : (r+1)\delta < 1\}, \qquad a = \frac{1 - 2\delta}{1 - \delta}. \tag{2.7}$$

The endpoint exclusion gives $D > K$ and $n - D > K$, so every index used below lies in the support and its corresponding mass is strictly positive. Recall that $c = D - 1$ is the rightmost modal index, and write $p = f_c = \max_k f_k$ for the maximal mass. Since $q_{D-1} \geq 1 > q_D$, the first inequality in (2.4) gives

$$\delta_{D-1} \leq \frac{\delta}{1-\delta}, \qquad q_D = q_{D-1}(1 - \delta_{D-1}) \geq a.$$

For $0 \leq j \leq K - 1$, (2.6) gives

$$\begin{aligned} q_{D+j} &= q_D \prod_{s=0}^{j-1} (1 - \delta_{D+s}) \\ &\geq a \prod_{s=0}^{j-1} \frac{1 - (s+1)\delta}{1 - s\delta} = a(1 - j\delta). \end{aligned}$$

For $1 \leq r \leq K$, we obtain the right normalized-mass lower bound

$$R_r := a^r \prod_{j=1}^{r-1} (1 - j\delta), \qquad \frac{f_{c+r}}{p} = \prod_{j=0}^{r-1} q_{D+j} \geq R_r. \tag{2.8}$$

For the left normalized-mass lower bounds, let $1 \leq j \leq K$. Then

$$q_D = q_{D-j} \prod_{s=1}^{j} (1 - \delta_{D-s}) < 1,$$

and hence

$$\begin{aligned} \frac{1}{q_{D-j}} &> \prod_{s=1}^{j} (1 - \delta_{D-s}) \\ &\geq \prod_{s=1}^{j} \frac{1 - (s+1)\delta}{1 - s\delta} = \frac{1 - (j+1)\delta}{1 - \delta}. \end{aligned}$$

For $1 \leq r \leq K$, we obtain the left normalized-mass lower bound

$$L_r := (1 - \delta)^{-r} \prod_{j=2}^{r+1} (1 - j\delta), \qquad \frac{f_{c-r}}{p} = \prod_{j=1}^{r} \frac{1}{q_{D-j}} \geq L_r. \tag{2.9}$$

3 Variance bounds and scalar reduction

Define auxiliary weights by $w_0 = 1$, $w_r = R_r$, and $w_{-r} = L_r$ for $1 \le r \le K$, and put

$$A(\delta) = \frac{1}{2} \sum_{i,j=-K}^{K} w_i w_j (i-j)^2. \tag{3.1}$$

The pairwise variance identity and the preceding normalized-mass lower bounds give

$$V = \frac{1}{2} \sum_{i,j=0}^{n} f_i f_j (i-j)^2 \ge p^2 A(\delta). \tag{3.2}$$

We also use the lattice maximal-mass bound of Bobkov, Marsiglietti, and Melbourne [3, Corollary 3.2]:

$$V \ge \frac{p^{-2}-1}{12}. \tag{3.3}$$

A short smoothing argument proves (3.3). Let U be independent of W and uniform on the unit interval centered at zero. The density of $W+U$ is bounded by p, and $\mathrm{Var}(W+U) = V + 1/12$. Let φ be any probability density bounded by p, and let $x_0 \in \mathbb{R}$. Its second moment about x_0 is minimized by filling the interval of length $1/p$ centered at x_0 at density p. Moving mass toward x_0 without exceeding the density bound lowers the integral until this density is reached. Hence

$$\int_{\mathbb{R}} (x-x_0)^2 \varphi(x)\, dx \ge p \int_{-1/(2p)}^{1/(2p)} t^2\, dt = \frac{1}{12p^2}.$$

Taking $x_0 = \mathbb{E}(W+U)$ yields $V + 1/12 \ge 1/(12p^2)$, proving (3.3).

It remains to prove one scalar estimate.

Proposition 3.1. *For $0 < \delta < 1/4$, the quantity in* (3.1) *satisfies*

$$A(\delta) \ge \frac{3+\delta}{4\delta^2}. \tag{3.4}$$

Proof of Theorem 1.1 from Proposition 3.1. Suppose that $V < 1/(4\delta)$. Inequality (3.3) then gives

$$p^2 > \frac{\delta}{3+\delta}.$$

On the other hand, (3.2) and (3.4) give

$$p^2 < \frac{1}{4\delta A(\delta)} \le \frac{\delta}{3+\delta},$$

a contradiction. □

4 The scalar inequality

Set

$$H = \frac{1-\delta}{\delta} = \frac{1}{\delta} - 1 > 3. \tag{4.1}$$

Under (4.1), the left normalized-mass lower bounds are

$$L_r = b_r := \prod_{s=1}^{r} \left(1 - \frac{s}{H}\right). \tag{4.2}$$

For $1 \le r \le K$, the right normalized-mass lower bounds satisfy $R_r \ge L_r = b_r$. Indeed, if $C_r = R_r/L_r$, then $C_1 = 1$ and, for $1 \le r < K$,

$$\frac{C_{r+1}}{C_r} = \frac{(H-1)(H+1-r)}{(H+1)(H-r-1)} = 1 + \frac{2r}{(H+1)(H-r-1)} \ge 1. \tag{4.3}$$

Since every auxiliary weight is nonnegative, the pairwise form in (3.1) is nondecreasing in each auxiliary weight. With K fixed,

$$\frac{\partial A}{\partial w_\ell} = \sum_{j=-K}^{K} w_j(\ell - j)^2 \ge 0.$$

Replacing every R_r by b_r doesn't increase the pairwise form; denote the resulting value by A_{sym}. For the symmetric auxiliary weights $w_0 = 1$ and $w_{\pm r} = b_r$, put

$$S = 1 + 2\sum_{r=1}^{K} b_r, \qquad T = 2\sum_{r=1}^{K} r^2 b_r. \tag{4.4}$$

The pairwise identity gives

$$A_{\mathrm{sym}} = \left(\sum_{r=-K}^{K} w_r\right)\left(\sum_{r=-K}^{K} r^2 w_r\right) - \left(\sum_{r=-K}^{K} r w_r\right)^2 = ST,$$

because the final sum vanishes by symmetry. Hence $A(\delta) \ge A_{\mathrm{sym}} = ST$. Under (4.1), the right-hand side of (3.4) is

$$Q(H) = \frac{(3H+4)(H+1)}{4}. \tag{4.5}$$

For $H \ge 4$, it remains to prove $ST \ge Q(H)$. On $3 < H < 4$, we instead retain the exact asymmetric normalized-mass lower bounds.

4.1 The compact range

On $3 < H \le 4$, use the exact normalized-mass lower bounds $L_1, L_2, L_3, R_1, R_2, R_3$. Direct simplification gives

$$A(\delta) - Q(H) = \frac{P(H)}{4H^5(H+1)^3}, \tag{4.6}$$

where

$$\begin{aligned} P(H) = {} & -3H^{10} - 16H^9 + 750H^8 - 3676H^7 + 6613H^6 - 5460H^5 \\ & + 800H^4 + 4696H^3 - 7176H^2 + 4272H - 912. \end{aligned}$$

After substituting $H = 3 + t$, the degree-ten Bernstein expansion of P on $0 \le t \le 1$ has eleven strictly positive rational coefficients.

For $m = 4, \dots, 15$ and $H \in [m, m+1]$, define

$$S_m = 1 + 2\sum_{r=1}^{m} b_r, \qquad T_m = 2\sum_{r=1}^{m} r^2 b_r, \tag{4.7}$$

and

$$P_m(H) = 4H^{2m}\big(S_m T_m - Q(H)\big). \tag{4.8}$$

Table 1: Compact-range certificate summary. The minimum is taken over the Bernstein coefficients of the stated cleared numerator; the final column gives the first sixteen hexadecimal characters of its coefficient-vector SHA-256 digest. Full coefficients and digests are in the supplement.

Cell	Degree	Count	Minimum coefficient	SHA-256 prefix
$[3,4]$ asym.	10	11	22272	`e1e452c9c93c30cd`
$[4,5]$	10	11	332800	`441f603ac3f60919`
$[5,6]$	12	13	476945150	`80a652bad6b13e69`
$[6,7]$	14	15	252843503616	`e3ad5553e47317b2`
$[7,8]$	16	17	141578177942152	`6c774572e048423c`
$[8,9]$	18	19	92316620768411648	`e9b192d522d49b1e`
$[9,10]$	20	21	71284663153149460650	`876f43f48d0c3c1e`
$[10,11]$	22	23	65053547560474378240000	`b741906966aed963`
$[11,12]$	24	25	69645362929101036447979796	`6c318dad110490486`
$[12,13]$	26	27	86706679760909016273411637248	`643c6c7cf85a5535`
$[13,14]$	28	29	124437022735783915561664544889462	`64f915da71a6e95d`
$[14,15]$	30	31	204172536352322626071425358548172800	`e31285e840aa69d9`
$[15,16]$	32	33	380093996939768323066638847260414000000	`0850f4a393088df0`

For $m < H \leq m+1$, we have $K = m$. At $H = m$, we have $K = m-1$ and $b_m = 0$. Hence $S_m = S$ and $T_m = T$ throughout the closed cell $[m, m+1]$. Each P_m is an integer polynomial of degree $2m+2$. After substituting $H = m+t$, all its full-degree Bernstein coefficients are strictly positive.

If $G(t) = \sum_{j=0}^{d} a_j t^j$ has degree d, the Bernstein conversion used here is

$$\beta_i = \sum_{j=0}^{i} a_j \frac{\binom{i}{j}}{\binom{d}{j}}, \qquad G(t) = \sum_{i=0}^{d} \beta_i \binom{d}{i} t^i (1-t)^{d-i}. \tag{4.9}$$

All polynomial identities and Bernstein coefficients in this step are exact and rational. The Bernstein expansions of $P(3+t)$ and the twelve polynomials $P_m(m+t)$, $4 \leq m \leq 15$, have, respectively, 11 and 264 strictly positive coefficients. Thus all 275 certificate coefficients are strictly positive.

The generator program reconstructs every identity before checking the signs. This reconstruction provides an internal consistency check. The accompanying full certificate data file records each exact power-basis numerator and all 275 reduced rational Bernstein coefficients. A separately implemented standard-library checker supplies a computational cross-check: it reconstructs the identities and Bernstein expansions for the thirteen compact scalar cells without importing the generator program. Because the Bernstein basis functions are nonnegative on $[0,1]$, the verified coefficient signs prove Proposition 3.1 for $3 < H \leq 16$.

4.2 The analytic range

Thirteen exact scalar-cell checks cover the compact range; one initial triangular cell and a uniform cell family cover the analytic range.

Suppose $H \geq 16$, and choose an integer $J \geq 5$ such that

$$\frac{J(J+1)}{2} \leq H \leq \frac{(J+1)(J+2)}{2}. \tag{4.10}$$

Either choice is valid at a shared endpoint. Since $\delta = (H+1)^{-1}$,

$$K = \max\{r \in \mathbb{Z}_{\geq 1} : r < H\}.$$

For $J \geq 5$, $J < J(J+1)/2 \leq H$, so $J \leq K$. For $1 \leq r \leq J$, the elementary product bound gives

$$b_r \geq \lambda_r := 1 - \frac{r(r+1)}{2H} \geq 0. \tag{4.11}$$

Put

$$S_0 = 1 + 2\sum_{r=1}^{J} \lambda_r = 2J + 1 - \frac{J(J+1)(J+2)}{3H}, \tag{4.12}$$

$$T_0 = 2\sum_{r=1}^{J} r^2\lambda_r = \frac{J(J+1)(2J+1)}{3} - \frac{M_4 + M_3}{H}, \tag{4.13}$$

where

$$M_3 = \frac{J^2(J+1)^2}{4}, \qquad M_4 = \frac{J(J+1)(2J+1)(3J^2+3J-1)}{30}.$$

Since $J \le K$, all omitted auxiliary weights are nonnegative and $b_r \ge \lambda_r$ for $r \le J$. Hence $S \ge S_0$, $T \ge T_0$, and $A(\delta) \ge ST \ge S_0T_0$. It remains to prove $S_0T_0 \ge Q(H)$.

Let $N_J(H) = H^2(S_0T_0 - Q(H))$. For $J = 5$, only $16 \le H \le 21$ is needed. After writing $H = 16 + 5t$, $0 \le t \le 1$, the degree-four Bernstein coefficients are

$$2360, \qquad 7500, \qquad \frac{25055}{2}, \qquad \frac{254205}{16}, \qquad \frac{31115}{2}.$$

For $J \ge 6$, write

$$H = \frac{J(J+1)}{2} + (J+1)t, \qquad J = u + 6, \qquad 0 \le t \le 1.$$

After removing strictly positive multipliers, the five degree-four Bernstein coefficients in t of

$$N_J\left(\frac{J(J+1)}{2} + (J+1)t\right)$$

are the following polynomials in u, in order:

$$\begin{aligned}
&121u^4 + 2474u^3 + 17431u^2 + 46014u + 25400,\\
&121u^5 + 3442u^4 + 37967u^3 + 199405u^2 + 480475u + 384510,\\
&121u^6 + 4410u^5 + 65863u^4 + 512764u^3 + 2172926u^2 + 4668316u + 3829440,\\
&121u^5 + 3684u^4 + 43735u^3 + 249961u^2 + 671859u + 644400,\\
&121u^4 + 2958u^3 + 25579u^2 + 88782u + 91440.
\end{aligned}$$

After factoring out the common strictly positive scalar $1/2880$, the removed multipliers are, respectively,

$$J^2(J+1)^2, \quad J(J+1)^2, \quad (J+1)^2, \quad (J+1)^2(J+2), \quad (J+1)^2(J+2)^2.$$

All five multipliers are strictly positive for $J \ge 6$, and every coefficient of the five polynomials above is strictly positive for $u \ge 0$. Thus $N_J(H) > 0$ and $S_0T_0 > Q(H)$ throughout the analytic range. Together with the compact range Bernstein verification, this proves Proposition 3.1.

Supplementary Material

Code and exact certificate data. The supplementary archive comprises the generator program, an independently implemented checker, the compact summary certificate, the full certificate, the MIT license, and a manifest giving the reference environment, exact replay commands, per-cell and aggregate digests, and archive status. The full certificate records the exact power-basis numerators and all 275 reduced rational Bernstein coefficients for the thirteen compact scalar cells. Using only the Python standard library and without importing the generator, the checker reconstructs the identities and Bernstein expansions, verifies strict positivity, and recomputes the Bernstein-vector digest for each cell, each cell-payload digest, and the overall payload digest. The programs verify only the thirteen compact scalar cells. Neither program machine-verifies Theorem 1.1. The archive also contains the complete conditional Lean project cited in the acknowledgement. That project formalizes the recurrence propagation, endpoint exclusion, first-crossing algebra, and raw-drop corollary, taking the normalized Hillion–Johnson recurrence as a hypothesis.

AI use and author responsibility. OpenAI's GPT-5.6 model, accessed through Codex, supplied substantial proof-search assistance. Other sessions with Claude (Anthropic), ChatGPT and Codex (both OpenAI), and Gemini (Google) assisted with computational exploration, exact-verification code, proof critique, formalization support, literature searching, drafting, and revision. Aristotle (Harmonic) completed bounded proof obligations in Lean for a conditional formalization; that formalization assumes the normalized Hillion–Johnson recurrence and doesn't formally prove Theorem 1.1. Its complete Lean project is included in the supplement. The author remains responsible for the mathematics, sources, wording, and released files. Fresh generator output matched both certificate files byte for byte, and the independently implemented checker replayed the full certificate in the reference environment; the exact commands and digests are listed in the supplement.

References

[1] Heshan Aravinda, *Entropy-variance inequalities for discrete log-concave random variables via degree of freedom*, Discrete Mathematics **347** (2024), no. 1, 113683.

[2] Jean-Bernard Baillon, Roberto Cominetti, and José Vaisman, *A sharp uniform bound for the distribution of sums of Bernoulli trials*, Combinatorics, Probability and Computing **25** (2016), no. 3, 352–361. MR3482659

[3] Sergey G. Bobkov, Arnaud Marsiglietti, and James Melbourne, *Concentration functions and entropy bounds for discrete log-concave distributions*, Combinatorics, Probability and Computing **31** (2022), no. 1, 54–72. MR4356456

[4] J. N. Darroch, *On the distribution of the number of successes in independent trials*, The Annals of Mathematical Statistics **35** (1964), no. 3, 1317–1321. MR164359

[5] Lutz Dümbgen and Jon A. Wellner, *The density ratio of Poisson binomial versus Poisson distributions*, Statistics & Probability Letters **165** (2020), 108862. MR4118941

[6] Alexander Gnedin, *Cross modality of the extended binomial sums*, 2024, arXiv:2408.06477 [math.PR].

[7] Erwan Hillion and Oliver Johnson, *Discrete versions of the transport equation and the Shepp–Olkin conjecture*, The Annals of Probability **44** (2016), no. 1, 276–306. MR3456338

[8] Oliver Johnson, *A discrete log-Sobolev inequality under a Bakry–Émery type condition*, Annales de l'Institut Henri Poincaré, Probabilités et Statistiques **53** (2017), no. 4, 1952–1970. MR3729642

[9] Arnaud Marsiglietti and James Melbourne, *Concentration inequalities for log-concave sequences*, International Mathematics Research Notices **2026** (2026), no. 4, rnag023.

[10] Jim Pitman, *Probabilistic bounds on the coefficients of polynomials with only real zeros*, Journal of Combinatorial Theory, Series A **77** (1997), no. 2, 279–303. MR1429082

[11] Wenpin Tang and Fengmin Tang, *The Poisson binomial distribution—old & new*, Statistical Science **38** (2023), no. 1, 108–119. MR4534645